Travelling and Periodic Wave Solutions of Some Nonlinear Wave Equations

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We present the mixed dn-sn method for finding periodic wave solutions of some nonlinear wave equations. Introducing an appropriate transformation, we extend this method to a special type of nonlinear equations and construct their solutions, which are not expressible as polynomials in the Jacobi elliptic functions. The obtained solutions include the well known kink-type and bell-type solutions as a limiting cases. Also, some new travelling wave solutions are found. – PACS: 02.30.Jr; 03.40.Kf

Key words: Nonlinear Wave Equation; Special Types of Nonlinear Equations; Periodic Wave Solutions; Travelling Wave Solutions.

1. Introduction

Travelling wave solutions of nonlinear partial differential equations (NLPDEs) describe important physical phenomena. Various methods, including the tanh-function method [1], the extended tanh-function method [2], the special truncated expansion method [3] and the homogeneous balance method [4] have been presented to find exact solutions of NLPDEs. Also, several ansatz equations [5 – 7] have been introduced to construct travelling and solitary wave solutions of nonlinear evolution equations (NLEEs). Travelling wave solutions of many NLPDEs can be expressed as polynomials of hyperbolic tanh and sech functions in using most of these methods. The symbolic software package to compute such solutions has been described in [8 – 10].

In the study of the Korteweg-de Vries (KdV) equation, the travelling wave solution leads to a periodic solution which is called cnoidal wave solution [11]. Periodic wave solutions of some NLEEs were recently obtained [12 – 15] in terms of Jacobi elliptic functions (JEFs). The JEF expansion method is used to construct periodic wave solutions of some nonlinear wave equations [12, 13]. It is a natural generalization of the tanh-function method for finding solitary wave solutions. Recently, the JEF method was expressed in a form suitable for automation and used to find periodic wave solutions to some NLEEs [14]. But physics often provides special types of nonlinear equations whose solutions cannot be expressed as polynomial solutions. Fan and Hon [16] have obtained multiple travelling wave solutions of such equations by using the generalized tanh-function method. Moreover, Fan and Zhang [17] have extended the JEF method and obtained doubly periodic wave solutions of a special-type of nonlinear equations.

The aim of this paper is to present the mixed dn-sn method and use it to obtain various periodic wave solutions of some nonlinear wave equations.

This paper is organized as follows. In Sect. 2 we describe the mixed dn-sn method to construct multiple periodic wave solutions of some nonlinear wave equations. In this method we introduce an ansatz equation which admits the dn (or nd)-function solution and must solve systems of algebraic equations. In Sect. 3 we illustrate this method by considering some nonlinear wave equations, such as the combined KdV and modified KdV equation, the modified Zakharov-Kuznetsov (mZK) equation, and the nondissipative ϕ^4- model equation. We also extend this method to the nonlinear Klein-Gordon equation and obtain several classes of periodic wave solutions. In the limiting case, the solitary and travelling wave solutions are also obtained. To show the properties of the obtained periodic solutions, we draw plots of these solutions. Finally, we conclude the paper in Section 4.

2. Sketch of the Mixed dn-sn Method

Consider a given NLPDE

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \quad (2.1)$$

Let $u(x,t) = u(\xi)$; $\xi = x - \omega t$, where ω is the wave speed, equation (2.1) may be reduced to an ordinary differential equation (ODE)

$$G(u, u', u'', \ldots) = 0, \quad u' = \frac{du}{d\xi}. \quad (2.2)$$

We search for the solution of the reduced ODE (2.2) in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{N} A_i W^i + \sqrt{a^2 - W^2} \sum_{i=0}^{N-1} b_i W^i, \quad (2.3)$$

where N is a positive integer which can be determined by comparing the behaviour of W^N in the highest derivatives with its counterpart within the nonlinear term(s) in (2.2), A_i and b_i are constants to be determined. We introduce $W = W(\xi)$ which satisfies the elliptic equation

$$W' = \sqrt{(a^2 - W^2)(W^2 - b^2)}, \quad a > b > 0. \quad (2.4)$$

The solutions of (2.4) are given by

$$\begin{aligned} W &= a\,\mathrm{dn}(a(x - \omega t)|m), \\ W &= a\sqrt{1-m}\,\mathrm{nd}(a(x - \omega t)|m), \end{aligned} \quad (2.5)$$

where a, b and m are related by $m = (a^2 - b^2)/a^2$, and $\mathrm{dn}(a\xi|m)$ and $\mathrm{nd}(a\xi|m) = 1/\mathrm{dn}(a\xi|m)$ are the JEFs with modulus m.

If $W = a\,\mathrm{dn}(a\xi)|m)$, then (2.3) becomes

$$u(x,t) = u(\xi) = \sum_{i=0}^{N} A_i a^i \mathrm{dn}^i(a\xi|m) + a\sqrt{m}\,\mathrm{sn}(a\xi|m) \sum_{i=0}^{N-1} b_i a^i \mathrm{dn}^i(a\xi|m), \quad (2.6)$$

while if $W = a\sqrt{1-m}\,\mathrm{nd}(a\xi)|m)$, then (2.3) becomes

$$u(x,t) = \sum_{i=0}^{N} A_i a^i (1-m)^{\frac{i}{2}} \mathrm{nd}^i(a\xi|m) + a\sqrt{m}\,\mathrm{cd}(a\xi|m) \sum_{i=0}^{N-1} b_i a^i (1-m)^{\frac{i}{2}} \mathrm{nd}^i(a\xi|m), \quad (2.7)$$

where $\mathrm{cd}(a\xi|m) = \mathrm{cn}(a\xi|m)/\mathrm{dn}(a\xi|m)$ and cn is the Jacobi cnoidal function. If $b_i = 0$, $i = 0, 1, 2, \ldots, N-1$, then (2.6) and (2.7) constitute the dn and nd expansions, respectively.

Substituting (2.3) into (2.2) yields the algebraic equation

$$P(W) + \sqrt{a^2 - W^2}\,Q(W) = 0,$$

where $P(W)$ and $Q(W)$ are polynomials in W. Setting the coefficients of the various powers of W in P and Q to zero will yield a system of algebraic equations in the unknowns A_i, b_i, a, ω and m. Solving this system, we can determine these unknowns. Therefore we can obtain several classes of periodic-wave solutions involving sn, dn and nd, cd functions. If a or any other parameter is left unspecified, then it is to be regarded as arbitrary for the solution of the NLPDE.

In the limit $m \to 1$, the JEFs $\mathrm{sn}(\xi|m)$ and $\mathrm{dn}(\xi|m)$ (or $\mathrm{cn}(\xi|m)$) degenerate into the hyperbolic functions $\tanh(\xi)$ and $\mathrm{sech}(\xi)$, respectively. Moreover, we can use the Jacobi transformations

$$\begin{aligned} \mathrm{dn}(a\xi|m) &= \mathrm{cn}(\sqrt{m}a\xi|m^{-1}), \\ \mathrm{sn}(a\xi|m) &= m^{-1/2}\,\mathrm{sn}(\sqrt{m}a\xi|m^{-1}) \end{aligned} \quad (2.8)$$

to compute solutions in terms of the functions cn and sn. Some more properties of JEFs are given in [18].

3. Applications

In this section we apply the mixed dn-sn method to some nonlinear wave equations whose balancing numbers are positive integers. Moreover, we construct the periodic wave solutions to a special-type of nonlinear equation whose solutions cannot be expressed in polynomial form.

3.1. The Combined KdV and Modified KdV Equation

We consider the combined KdV and modified KdV equation [11, 20]

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u_{xxx} = 0, \quad \beta \neq 0, \quad (3.1)$$

where α, β and δ are constants. Equation (3.1) is widely used in various fields, such as quantum field theory, solid-state physics, plasma physics and fluid physics [11, 19, 20]. Let $u = u(\xi)$ and (3.1) be transformed to the reduced equation

$$-\omega u' + \alpha u u' + \beta u^2 u' + \delta u''' = 0. \quad (3.2)$$

Balancing u''' with $u^2 u'$ yields $N = 1$, so we may choose

$$u(x,t) = A_0 + A_1 W + b_0 \sqrt{a^2 - W^2}. \quad (3.3)$$

Substituting (3.3) into (3.2) we obtain the following set of algebraic equations:

$$[\beta A_1^2 - 3\beta b_0^2 - 6\delta]A_1 = 0,$$
$$[3\beta A_1^2 - \beta b_0^2 - 6\delta]b_0 = 0,$$
$$(\alpha + 2\beta A_0)(A_1^2 - b_0^2) = 0,$$
$$(\alpha + 2\beta A_0)A_1 b_0 = 0,$$
$$[-\omega + \alpha A_0 + \beta A_0^2 + \beta b_0^2 a^2 + \delta(a^2 + b^2)]A_1 = 0,$$
$$b^2 = a^2(1-m),$$
$$[-\omega + \alpha A_0 + \beta A_0^2 - 8\beta a^2 A_1^2 + 3\beta b_0^2 a^2 + \delta(16a^2 + b^2)]b_0 = 0,$$

which has the solutions

$$A_0 = -\frac{\alpha}{2\beta}, \quad A_1 = \pm\sqrt{6\delta/\beta}, \quad b_0 = 0, \quad \omega = -\frac{\alpha^2 - 4\beta\delta a^2(2-m)}{4\beta}, \tag{3.4}$$

$$A_0 = -\frac{\alpha}{2\beta},\ A_1 = 0,\ b_0 = \pm\sqrt{-6\delta/\beta}, \quad \omega = -\frac{\alpha^2 + 4\beta\delta a^2(m+1)}{4\beta}, \tag{3.5}$$

$$A_0 = -\frac{\alpha}{2\beta},\ A_1 = \pm\sqrt{3\delta/2\beta},\ b_0 = \pm i\sqrt{3\delta/2\beta}, \quad \omega = -\frac{\alpha^2 - 2\beta\delta a^2(1-2m)}{4\beta}. \tag{3.6}$$

Substituting (3.4) – (3.6) into (3.3) and using the special solutions (2.5) of equation (2.4), we obtain the following classes of periodic wave solutions of (3.1):

$$u = -\frac{\alpha}{2\beta} \pm a\sqrt{6\delta/\beta}\ \mathrm{dn}(a(x-\omega t)|m), \quad u = -\frac{\alpha}{2\beta} \pm a\sqrt{6\delta(1-m)/\beta}\ \mathrm{nd}(a(x-\omega t)t)|m), \tag{3.7}$$

where ω is give by (3.4),

$$u = -\frac{\alpha}{2\beta} \pm a\sqrt{-6m\delta/\beta}\ \mathrm{sn}\Big\{a\Big[x + \frac{\alpha^2 + 4\beta\delta a^2(m+1)}{4\beta}t\Big]|m\Big\}, \tag{3.8}$$

$$u = -\frac{\alpha}{2\beta} \pm a\sqrt{-6m\delta/\beta}\ \mathrm{cd}\Big\{a\Big[x + \frac{\alpha^2 + 4\beta\delta a^2(m+1)}{4\beta}t\Big]|m\Big\}, \tag{3.9}$$

with a and m arbitrary and

$$u = -\frac{\alpha}{2\beta} \pm a\sqrt{3\delta/2\beta}\ \big[\mathrm{dn}\,(a(x-\omega t)|m) \pm i\sqrt{m}\,\mathrm{sn}\,(a(x-\omega t)|m)\big],$$
$$u = -\frac{\alpha}{2\beta} \pm a\sqrt{3\delta/2\beta}\ \Big[\sqrt{1-m}\,\mathrm{nd}\,(a(x-\omega t)|m) \pm i\sqrt{m}\,\mathrm{cd}\,(a(x-\omega t)|m)\Big], \tag{3.10}$$

where a, m are arbitrary and ω is given by (3.6). If we put $\alpha = 0$ in (3.8), we get a periodic solution of the modified KdV equation which coincides with that given by Liu et al. [12]. Moreover, the solutions (3.8) and (3.9) to (3.1) given in [15] are recovered. With $m \to 1$ in (3.7) – (3.10), the solitary wave solutions to (3.1) given in [10, 15, 20] are also recovered.

To show the properties of the periodic wave solutions to the combined KdV and modified KdV equation, we draw the plots of the obtained solutions (3.7) and (3.10) and their positions at $t = 0$ with $\alpha = 1$, $\beta = 0.5$, $\delta = 1$, $a = 1$, $m = 0.25$ (see Fig. 1).

3.2. The mZK Equation

The equation

$$u_t + \beta u^2 u_x + u_{xxx} + u_{yyx} = 0 \tag{3.11}$$

is the mZK in (2+1) dimensions, which is a model for acoustic plasma waves [21]. We put its ansatz solutions as in (3.3), but with $\xi = x + ly - \omega t$, l being constant. The mixed dn-sn method gives the following solutions of (3.11):

$$u = \pm a\sqrt{6(1+l^2)/\beta}\ \mathrm{dn}(a\xi|m),$$

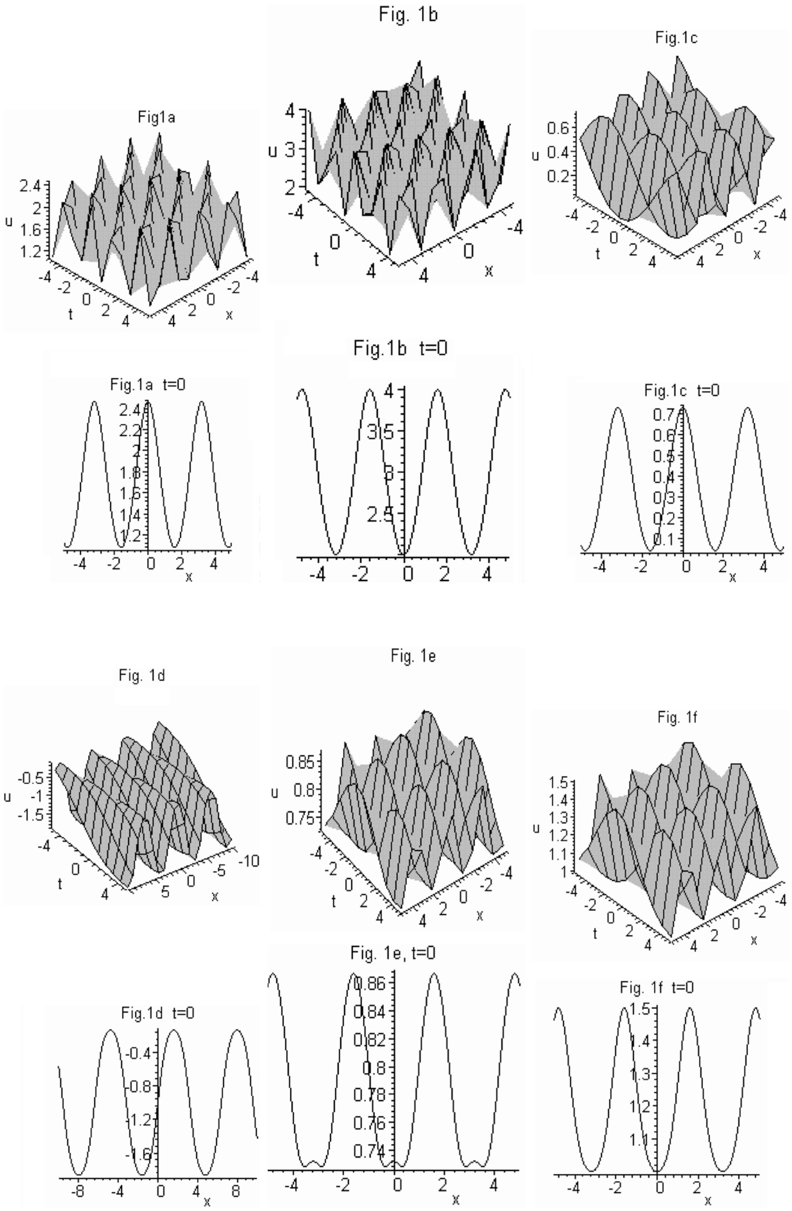

Fig. 1. The solution (3.7) and its position at $t = 0$; Fig. 1a in terms of the function dn; Fig. 1b in terms of the function nd. The periodic wave solution (3.10) in terms of the functions dn, sn and its position at $t = 0$; Fig. 1c the real part; Fig. 1d the imaginary part; Fig. 1e the modulus. The modulus of the solution (3.10) in terms of the functions nd, cd and its position at $t = 0$, Fig. 1f.

$$u = \pm a\sqrt{6(1+l^2)(1-m)/\beta}\ \mathrm{nd}(a\xi|m),$$

$$\omega = (1+l^2)(2-m)a^2, \quad (3.12)$$

$$u = \pm a\sqrt{-6m(1+l^2)/\beta}\ \mathrm{sn}(a\xi|m),$$

$$u = \pm a\sqrt{-6m(1+l^2)/\beta}\ \mathrm{cd}(a\xi|m),$$

$$\omega = -(1+l^2)(m+1)a^2, \quad (3.13)$$

a and m being arbitrary and

$$u = \pm a\sqrt{3(1+l^2)/2\beta}\ [\mathrm{dn}(a\xi|m) \pm i\sqrt{m}\,\mathrm{sn}(a\xi|m)],$$

$$u = \pm a\sqrt{3(1+l^2)/2\beta}\ [\sqrt{1-m}\,\mathrm{nd}(a\xi|m) \pm i\sqrt{m}\,\mathrm{cd}(a\xi|m)], \quad (3.14)$$

with $\omega = \frac{1}{2}(1+l^2)(1-2m)a^2$.

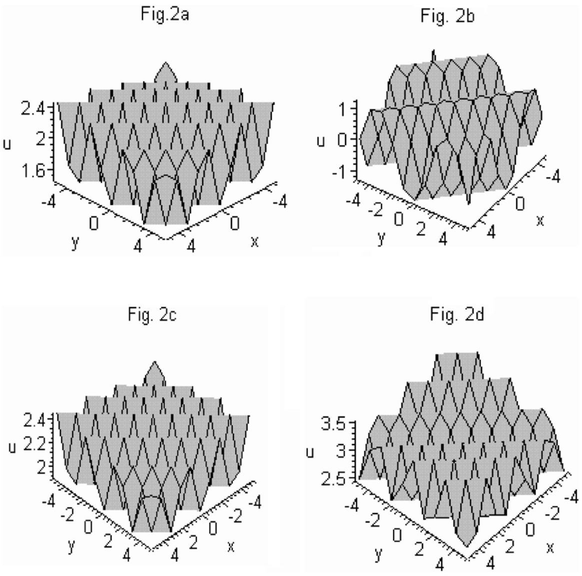

Fig. 2. The periodic wave solution (3.14) at $t = 0$ in terms of the functions dn, sn with $\beta = 0.5$, $l = 1$, $a = 1$, $m = 0.25$; Fig. 2a the real part; Fig. 2b the imaginary part; Fig. 2c the modulus. The modulus of periodic wave solution (3.14) in terms of the functions nd, cd with $t = 0$, $\beta = 0.5$, $l = 1$, $a = 1$, $m = 0.25$, Fig. 2d.

When $m \to 1$, then (3.12)–(3.14) become the solitary wave solutions of (3.11)

$$u = \pm k\sqrt{6(1+l^2)/\beta}\,\operatorname{sech}(k\xi),\ \omega = (1+l^2)k^2,$$

$$u = \pm k\sqrt{-6(1+l^2)/\beta}\,\tanh(k\xi),\ \omega = -2(1+l^2)k^2,$$

$$u = \pm k\sqrt{3(1+l^2)/2\beta}\,\big[\operatorname{sech}\big(k(x+ly-\omega t)\big) \pm i\tanh\big(k(x+ly-\omega t)\big)\big],$$

$$\omega = -\frac{1}{2}(1+l^2)k^2,$$

where k is an arbitrary constant.

The plots for the periodic wave solutions (3.14) at $t = 0$ are given in Figure 2.

3.3. The Nondissipative ϕ^4-model Equation

We consider the nondissipative ϕ^4-model equation

$$u_{tt} - c^2 u_{xx} + \alpha u - \beta u^3 = 0, \qquad (3.15)$$

where c, α and β are real constants. It is a nonlinear Klein-Gordon equation with cubic nonlinearity. Equation (3.15) can be used as a model field theory [22, 23].

Similarly, the mixed dn-sn method gives the solution in the form (3.3). Inserting (3.3) into (3.15) and determining the unknowns, we obtain the following multiple periodic wave solutions of (3.15):

$$u = \pm\sqrt{\frac{2\alpha}{\beta(2-m)}} \cdot \operatorname{dn}\left(\sqrt{\frac{\alpha}{(2-m)(c^2-\omega^2)}}\,(x-\omega t)|m\right),$$
$$u = \pm\sqrt{\frac{2\alpha(1-m)}{\beta(2-m)}} \cdot \operatorname{nd}\left(\sqrt{\frac{\alpha}{(2-m)(c^2-\omega^2)}}\,(x-\omega t)|m\right), \qquad (3.16)$$

$$u = \pm\sqrt{\frac{2m\alpha}{\beta(m+1)}} \cdot \operatorname{sn}\left(\sqrt{\frac{\alpha}{(m+1)(\omega^2-c^2)}}\,(x-\omega t)|m\right), \qquad (3.17)$$

$$u = \pm\sqrt{\frac{2m\alpha}{\beta(m+1)}} \cdot \operatorname{cd}\left(\sqrt{\frac{\alpha}{(m+1)(\omega^2-c^2)}}\,(x-\omega t)|m\right), \qquad (3.18)$$

and

$$u = \mp\sqrt{\alpha/2\beta(1-2m)} \cdot [\operatorname{dn}(a(x-\omega t)|m) + i\sqrt{m}\operatorname{sn}(a(x-\omega t)|m)], \qquad (3.19)$$

$$u = \mp\sqrt{\alpha/2\beta(1-2m)} \cdot [\sqrt{1-m}\,\operatorname{nd}(a(x-\omega t)|m) + i\sqrt{m}\,\operatorname{cd}(a(x-\omega t)|m)], \qquad (3.20)$$

with $a = \sqrt{\alpha/(2m-1)(\omega^2-c^2)}$, $\omega^2 \neq c^2$ and ω, m are arbitrary. Expression (3.17) for u was given by (40) in [12]. The complex conjugates of (3.19) and (3.20) are also solutions.

In the limit case $m \to 1$, (3.16), (3.17) and (3.19) become the travelling wave solutions of (3.15):

$$u = \pm\sqrt{2\alpha/\beta}\,\operatorname{sech}\big[\sqrt{\alpha/(c^2-\omega^2)}\,(x-\omega t)\big],$$
$$u = \pm\sqrt{\alpha/\beta}\,\tanh\big[\sqrt{\alpha/2(\omega^2-c^2)}\,(x-\omega t)\big], \qquad (3.21)$$

$$u = \pm\sqrt{\frac{-\alpha}{2\beta}}\left\{\operatorname{sech}\left[\sqrt{\frac{\alpha}{\omega^2-c^2}}(x-\omega t)\right] + i\tanh\left[\sqrt{\frac{\alpha}{\omega^2-c^2}}\,(x-\omega t)\right]\right\}. \qquad (3.22)$$

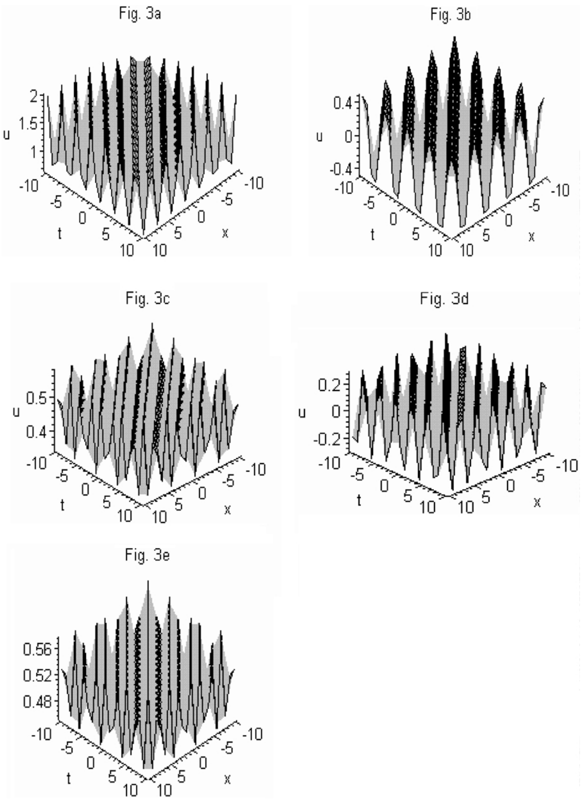

Fig. 3. The solution (3.16) in terms of the function nd with $\omega = 1$, $m = 0.5$ and $c = 1.25$, Fig. 3a. The solution (3.18) with $\omega = 1$, $m = 0.5$ and $c = 0.8$, Fig. 3b. The solution (3.19) with $\omega = 1$, $m = 0.25$ and $c = 1.25$; Fig. 3c the real part; Fig. 3d the imaginary part; Fig. 3e the modulus.

The properties of the periodic wave solutions (3.16), (3.18) and (3.19), with the parameters $\alpha = 1$, $\beta = 3$ are shown in Figure 3.

In the following we consider the nonlinear Klein-Gordon equation which cannot be directly solved by our methods. However, introducing appropriate transformations, we show that the method described in Sect. 2 can also be apply to such equation.

3.4. The Nonlinear Klein-Gordon Equation

Consider the nonlinear Klein-Gordon equation

$$u_{tt} - c^2 u_{xx} + \alpha u - \beta u^3 + \delta u^5 = 0, \tag{3.23}$$

where c, α, β and δ are real constants which satisfy $12\beta^2 < 64\alpha\delta < 15\beta^2$. This equation is a higher-order approximation of the standard ϕ^4-model widely used in the field theory [22]. Obviously, (3.23) has the balancing number $N = 1/2$. Therefore, we make the transformation

$$v = u^2, \quad v = V(\xi), \quad \xi = x - \omega t$$

and change (3.23) into the form

$$(\omega^2 - c^2)[2VV'' - (V')^2] + 4(\alpha V^2 - \beta V^3 + \delta V^4) = 0. \tag{3.24}$$

Now balancing VV'' and V^4 in (3.24), we find $N = 1$. In this case we can assume that

$$V = A_0 + A_1 W + b_0\sqrt{a^2 - W^2}. \tag{3.25}$$

Inserting (3.25) into (3.24) and determining the unknowns, we obtain several types of periodic wave solutions as

$$\begin{aligned} u &= \frac{1}{2}\sqrt{\frac{3\beta}{2\delta}}\left[1 \pm \mathrm{dn}\left(\frac{3\beta}{4\sqrt{3\delta(\omega^2 - c^2)}}(x - \omega t)|m\right)\right]^{\frac{1}{2}}, \quad m = \frac{4(16\alpha\delta - 3\beta^2)}{3\beta^2}, \\ u &= \frac{1}{2}\sqrt{\frac{3\beta}{2\delta}}\left[1 \pm \sqrt{\frac{15\beta^2 - 64\alpha\delta}{3\beta^2}}\mathrm{nd}\left(\frac{3\beta}{4\sqrt{3\delta(\omega^2 - c^2)}}(x - \omega t)|m\right)\right]^{\frac{1}{2}}, \quad m = \frac{4(16\alpha\delta - 3\beta^2)}{3\beta^2}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} u &= \left[\frac{3\beta}{8\delta} \pm \frac{\sqrt{3(15\beta^2 - 64\alpha\delta)}}{8\delta}\mathrm{sn}\left(\frac{3\beta}{4\sqrt{3\delta(c^2 - \omega^2)}}\xi|m\right)\right]^{\frac{1}{2}}, \\ u &= \left[\frac{3\beta}{8\delta} \pm \frac{\sqrt{3(15\beta^2 - 64\alpha\delta)}}{8\delta}\mathrm{cd}\left(\frac{3\beta}{4\sqrt{3\delta(c^2 - \omega^2)}}\xi|m\right)\right]^{\frac{1}{2}}, \quad m = \frac{15\beta^2 - 64\alpha\delta}{3\beta^2}, \end{aligned} \tag{3.27}$$

$$u=\sqrt{\frac{3\beta}{8\delta}}\left[1\pm\left(\frac{15\beta^2-64\alpha\delta}{3\beta^2}\right)^{\frac{1}{4}}\left(\mathrm{dn}(a\xi|m)+i\sqrt{m}\,\mathrm{sn}(a\xi|m)\right)\right]^{\frac{1}{2}},$$
$$u=\sqrt{\frac{3\beta}{8\delta}}\left[1\pm\left(\frac{15\beta^2-64\alpha\delta}{3\beta^2}\right)^{\frac{1}{4}}\left(\sqrt{1-m}\,\mathrm{nd}(a\xi|m)+i\sqrt{m}\,\mathrm{cd}(a\xi|m)\right)\right]^{\frac{1}{2}}, \tag{3.28}$$

where $a=\sqrt{\frac{\beta\sqrt{3(15\beta^2-64\alpha\delta)}}{4\delta(\omega^2-c^2)}}$, $\omega^2\neq c^2$, $m=\frac{1}{2}[1-\frac{9\beta^2-32\alpha\delta}{\beta\sqrt{3(15\beta^2-64\alpha\delta)}}]$; and

$$u=\sqrt{\frac{3\beta}{8\delta}}\left[1\pm\sqrt{\frac{64\alpha\delta-15\beta^2}{\beta\sqrt{3(15\beta^2-64\alpha\delta)}}}(\mathrm{dn}(a\xi|m)+i\sqrt{m}\,\mathrm{sn}(a\xi|m))\right]^{\frac{1}{2}},$$
$$u=\sqrt{\frac{3\beta}{8\delta}}\left[1\pm\sqrt{\frac{64\alpha\delta-15\beta^2}{\beta\sqrt{3(15\beta^2-64\alpha\delta)}}}(\sqrt{1-m}\,\mathrm{nd}(a\xi|m)+i\sqrt{m}\,\mathrm{cd}(a\xi|m))\right]^{\frac{1}{2}}, \tag{3.29}$$

with $a=\sqrt{\frac{\beta\sqrt{3(15\beta^2-64\alpha\delta)}}{4\delta(c^2-\omega^2)}}$ and $m=\frac{1}{2}[1+\frac{9\beta^2-32\alpha\delta}{\beta\sqrt{3(15\beta^2-64\alpha\delta)}}]$.

The complex conjugates of (3.28) and (3.29) are also solutions. When $m>1$, the solution (3.29) may be transformed into an equivalent one by using of the Jacobi transformations (2.8).

When $3\beta^2=16\alpha\delta$, we obtain the following classes of travelling wave solutions of (3.23):

$$u=\left(\frac{3\alpha}{4\delta}\right)^{\frac{1}{4}}\left[1\pm\tanh\sqrt{\frac{\alpha}{c^2-\omega^2}}(x-\omega t)\right]^{\frac{1}{2}}, \tag{3.30}$$

$$u=\sqrt{\frac{3\beta}{8\delta}}\left[1+i\,\mathrm{sech}(2\sqrt{\frac{\alpha}{c^2-\omega^2}}\xi)\mp\tanh\left(2\sqrt{\frac{\alpha}{c^2-\omega^2}}\xi\right)\right]^{\frac{1}{2}}, \tag{3.31}$$

in addition to its the two complex conjugates. The travelling wave solutions (3.30) to (3.23) given in [6] are recovered.

Plots for the periodic wave solutions (3.26), and (3.27) with $\alpha=0.2$, $\beta=\delta=1$ are given in Figure 4.

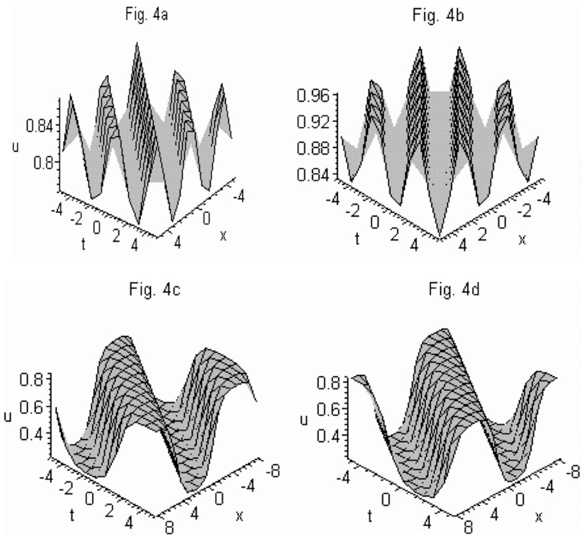

Fig. 4. The solutions (3.26) with $\omega=1$ and $c=0.8$; Fig. 4a in terms of the function dn; Fig. 4b in terms of the function nd. The solutions (3.27) with $\omega=1$ and $c=1.25$; Fig. 4c in terms of the function sn; Fig. 4d in terms of the function cd.

4. Conclusion

We have suggested the mixed dn-sn method and used it to construct multiple periodic wave solutions for a variety of nonlinear wave equations. We applied this method to the combined KdV and modified KdV equation, the mZK equation and the nondissipative ϕ^4- model equation, and obtained their solutions. Through introducing an appropriate transformation, it is shown that this method is also applicable to the nonlinear Klein-Gordon equation (3.23). On using the mixed dn-sn method we recovered not only the known solutions of (3.1) and (3.15) given in [12], but also found new solutions of such equations with no extra effort. Our method is more general than the dn-function method [24], the sech method and the tanh method and may be applied to other nonlinear evolution equations of mathematical physics. Moreover, several classes of travelling wave solutions of the considered equations can be constructed from the obtained periodic wave solutions as reduced case ($m \to 1$).

The obtained solutions include Jacobi doubly periodic wave solutions and soliton solutions. The properties of the periodic wave solutions are shown in Figures 1 – 4. Since the considered equations have been shown to be applicable to many dynamic problems in physics and other fields, the new doubly periodic solutions found here may be relevant in those subjects and fields of studies.

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